VL 15.11.2019

① Differentialgleichungen / gewöhnliche (ODE)

deterministic, first order ↳ partielle?

$$\dot{x} = f(x, t, p) \;\hat{=}\; \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

time, parameters

rate of change

autonomous: $\cancel{\dot{x}} = f(x, p)$, no t

② Wiederholung: 1 dim ODE

$\dot{x} = f(x, p)$

for example: $\dot{x} = kx - ax^2$ [Verhulst 1836]

continuous version of logistic map

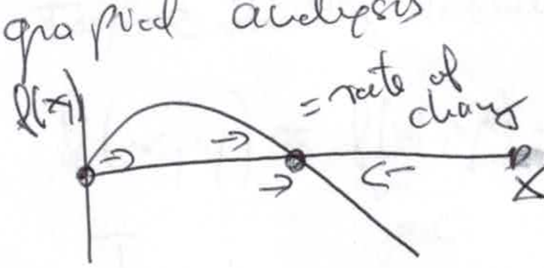
modelling population growth

also $\dot{x} = k \cdot x \left(1 - \frac{a}{k} x\right)$

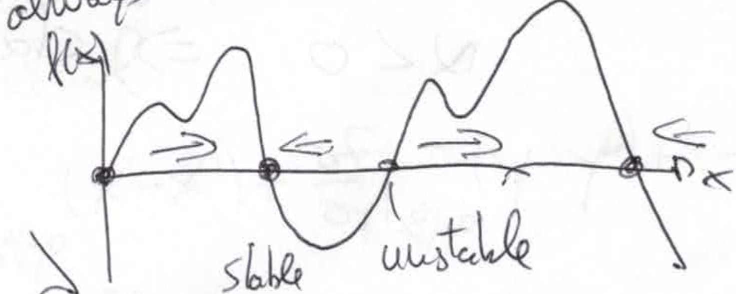
③ Fixpoints & stability

1 dim: $f(x^0, p) = 0$

graphical analysis



always works!



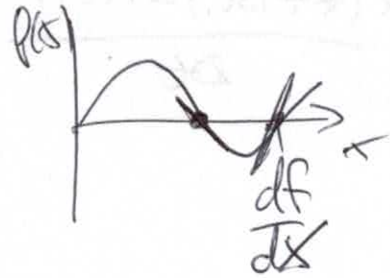
↳ discuss changing parameter

Was passiert wenn man Parameter ändert?

formally:

Formally:

derivative at the fix point



taylor series

$$\dot{x} = f(x) \underset{\substack{\uparrow \\ \text{am} \\ \text{punkt } x^0}}{\approx} f(x^0) + \frac{\partial f}{\partial x}\Big|_{x^0} (x - x^0) + \underbrace{\mathcal{O}((x-x^0)^2)}_{\frac{1}{2} f''(x-x^0)^2}$$

$$\Delta x = x - x^0$$

$$\Rightarrow \dot{\Delta x} = \dot{x} = \underbrace{f(x^0)}_{=0} + \frac{\partial f}{\partial x}\Big|_{x^0} \Delta x + \mathcal{O}(\Delta x^2)$$

linear equation $\dot{\Delta x} = \alpha \cdot \Delta x$

$$\Delta x(t) = e^{\alpha t} \cdot \Delta x_0(0)$$

$\alpha > 0 \Rightarrow$ exp. growth

$\alpha < 0 \Rightarrow$ stability

2 dimensions

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} = \vec{F}(x,y)$$

Beispiel: Lotka - Volterra

$$\dot{x} = \alpha x - \beta x y$$
$$\dot{y} = \gamma \cancel{x} y - \delta y$$

} simple population model / Räuber Beute Modell

(1!) nullcline / isocline

↓ derivative is 0

↳ derivative is same value

↳ population dynamics rate of change of population is zero

$f_x = 0$ x-nullcline

$f_y = 0$ y-nullcline

fixpoint ⟺ both are zero !

phaseplane

(4b) formal analysis: reduction to linear model

Taylorentwicklung mehrdimensional

$$f(x,y) \approx \underbrace{f(x^0,y^0)}_{=0} + \frac{\partial f}{\partial x}\Big|_{x^0,y^0}(x-x^0) + \frac{\partial f}{\partial y}\Big|_{x^0,y^0}(y-y^0) + \dots$$

↳ für f_x und f_y

Linearisierung der Form $\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \underbrace{\begin{bmatrix} f_x(x^0,y^0) \\ f_y(x^0,y^0) \end{bmatrix}}_{=0} + J \cdot \begin{pmatrix} x-x^0 \\ y-y^0 \end{pmatrix}$

⑤ Jacobi-Matrix am Fixpunkt ~~x^0, y^0~~ x^0, y^0 $\quad x^0 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$

$$\cancel{J_{ij} = \frac{\partial f_i}{\partial x_j}\Big|_{x^0}}$$

$$J = \begin{pmatrix} \frac{\partial f_x}{\partial x}\Big|_{x^0,y^0} & \frac{\partial f_x}{\partial y}\Big|_{x^0,y^0} \\ \frac{\partial f_y}{\partial x}\Big|_{x^0,y^0} & \frac{\partial f_y}{\partial y}\Big|_{x^0,y^0} \end{pmatrix}$$

allgemein: fixpunkt $x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}$ $\quad \dot{x} = \vec{f}(x)$

$$J_{ij} = \frac{\partial f_i}{\partial x_j}\Big|_{x^0}$$

"Funktionalmatrix"

benannt nach Carl Gustav Jacob Jacobi 1804–1851 Univ. Berlin

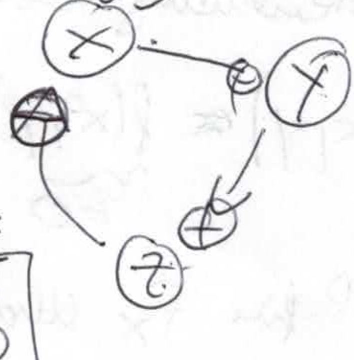
"das system"
welche variable hängt von welcher ab?
und mit welchem Vorzeichen

$\dot{x} = f(x, z) = \alpha z - \delta x$

$\dot{y} = f(y, x) = \beta x - \delta y$

$\dot{z} = (y, z) = \gamma y - \delta z$

$$J = \begin{array}{c|ccc} & x & y & z \\ \hline f_x & -\delta & 0 & -\gamma \\ f_y & +\beta & -\delta & 0 \\ f_z & 0 & +\gamma & -\delta \end{array}$$



In words: analyse dynamics nur anhand von J

$$\frac{\partial f_x}{\partial y}\Big|_{x^0} = ?$$

(6) Stabilität → nichtlineare System

→ zurück auf lineare System

$\dot{\Delta x} = J(x^0)\cdot \Delta x$ mehrdim. lineare ODE

Lösung hängt ab von den Eigenwerte der J-Matrix

$(\cancel{J - \lambda \hat{e}})\cdot x = 0$

$J x = \lambda x$

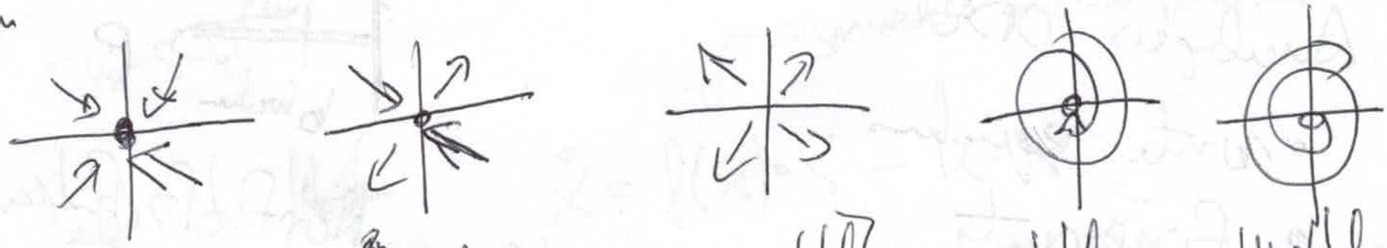
~~alle real Eigenwerte EV~~ < 0

⇒ Eigenwerte sind komplexe Zahlen

Re(EV) < 0 ⇒ Fixpunkt stabil

Im(EV) ≠ 0 ⇒ oscillationen

2dim



stable node — Re(EV) < 0

saddle (unstable) — max[Re(EV)] > 0

unstable node

stable focus — Re(EV) < 0, Im(EV) ≠ 0

unstable focus — Re(E) > 0, Im(E) ≠ 0

Klausur Garantie:

Was gibt es für Möglichkeiten, und wie nennt man sie?

Bifurkation!

grafisch

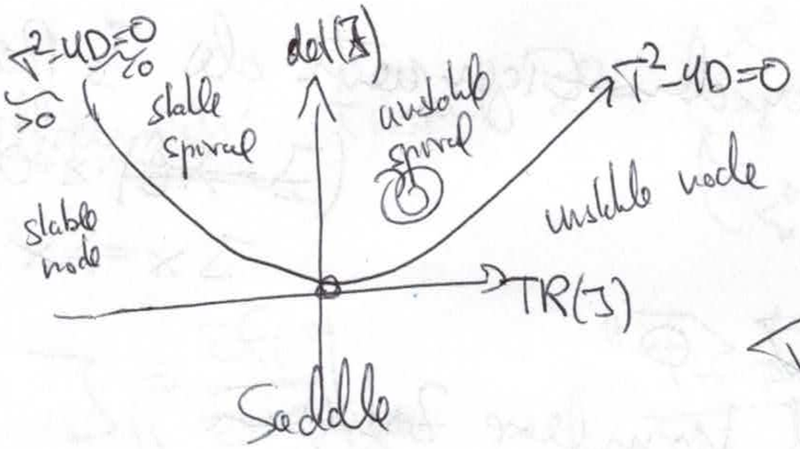
$$J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\det(\underline{J} - \lambda \underline{E}) = 0$$

charakteristisches Polynom

für 2x2 Matrizen: $T = A + A$

$D = AD - BC$



$$\lambda_{1/2} = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4D}\right]$$

Transitions are called Bifurkations

↓

Übergang

Hopf-Bifurkation

fixpunkt EV

überquert imag achse

stabil → instabil

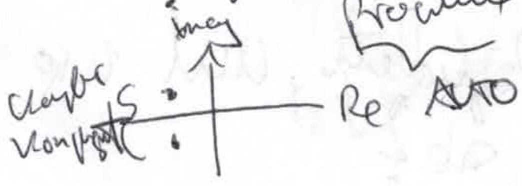
Sattel-Knoten

1 EV überquert die achse

Handwerkszeug:

Analysis ODE:

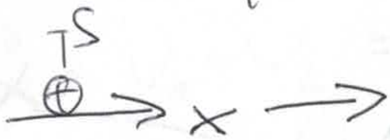
- write equations
- fixpoints
- phaseplane analyse (if possible)
- stability analysis
- Bifurkationsdiagramm



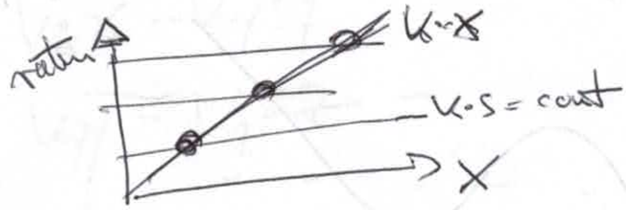
Beispiele: Dynamik von einfachen Signalwegen

grafische Analyse

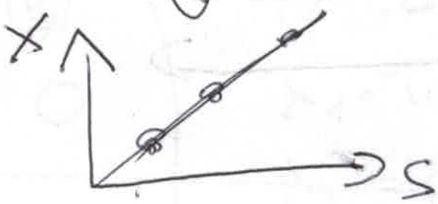
(1)



$$\dot{X} = \overbrace{K \cdot S}^{\text{rate}} - \overbrace{K_x \cdot X}^{\text{rate}} = f(X)$$

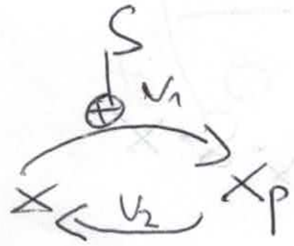


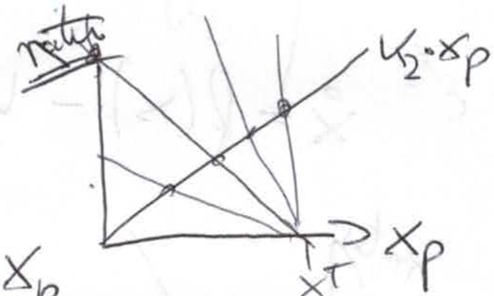
$$\left[X^0 = \frac{K_s}{K_x} \cdot S \right]$$

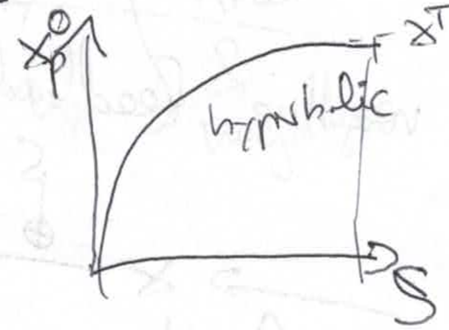


linear response

(2)





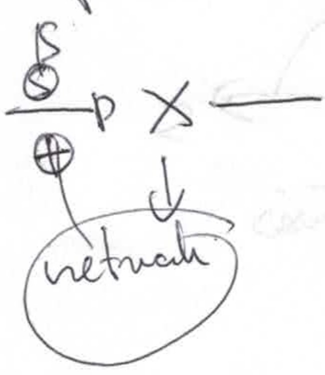


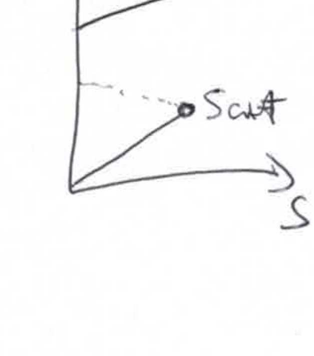
$$\dot{X}_p = K_1 \cdot S \cdot X - K_2 \cdot X_p$$
$$= K_1 \cdot S (X^T - X_p)$$
$$= K_1 \cdot S \cdot X^T$$

sigmoidale aktivierungs funktion

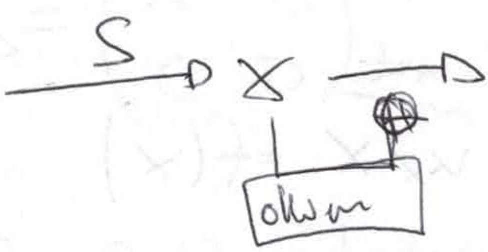
(3) self activation

$$\dot{X} = f(X) \cdot S - K \cdot X$$

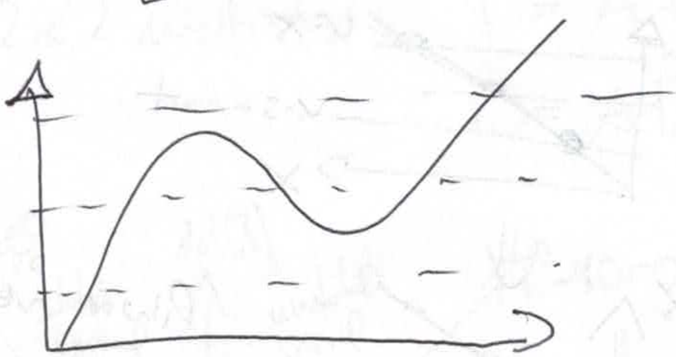


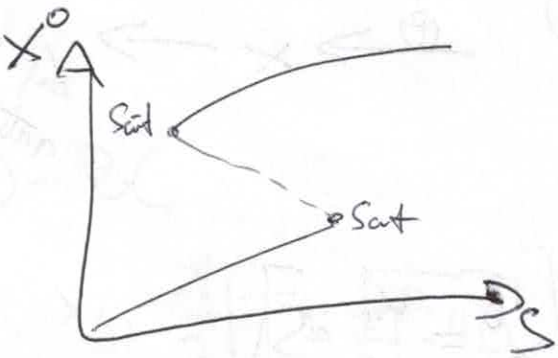


ahhung des Abbaus



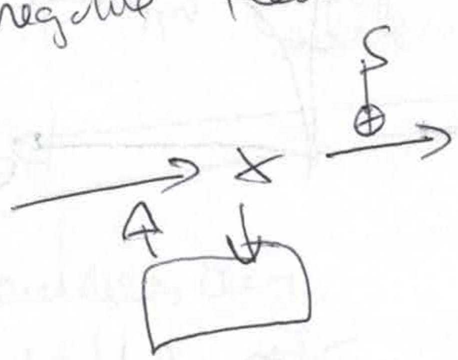
$$\dot{x} = k_s \cdot S - \underbrace{f(x)}\cdot X$$

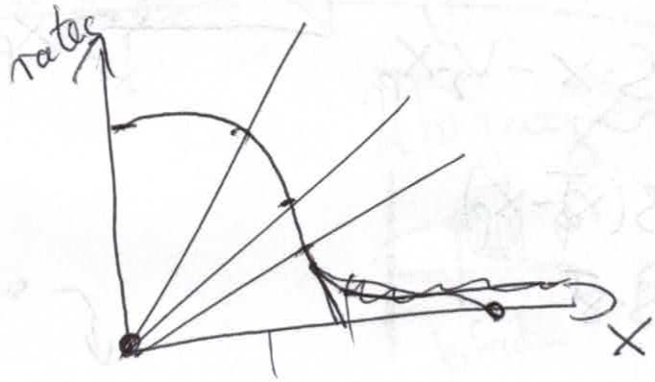




negative feedback

$$\dot{x} = f(x) - k_x \cdot S \cdot X$$





homeostasis

…odwin-Oscillato

$$\xrightarrow{V_1} X \xrightarrow{V_2} Y \xrightarrow{V_3} Z \xrightarrow{V_4}$$

$f(Z) \nearrow = \text{const}$

$\rightarrow f = f(Z)$

$$\dot X = V_1 - V_2 = f(Z) - k_x \cdot X$$

$$\dot Y = V_2 - V_3 = k_x X - k_y \cdot Y$$

$$\dot Z = V_3 - V_2 = k_y Y - k_z \cdot Z$$

$$J = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} f_x \\ f_y \\ f_z \end{array} & \begin{pmatrix} -k_x & 0 & \boxed{y_z} \\ k_x & -k_y & 0 \\ 0 & k_y & -k_z \end{pmatrix} \end{array}$$

$f = \text{const}$

$$\left.\begin{array}{l} \lambda_1 = -k_x \\ \lambda_2 = -k_y \\ \lambda_3 = -k_z \end{array}\right\} \text{all real \& negative}$$