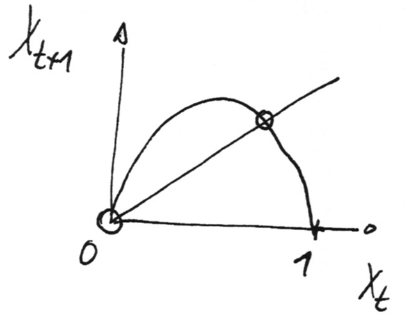
Differentialgleichungen / ODE
(ordinary differential equations)

→ main theme of lecture

Diskrete Abbildungen:

$$X_{t+1} = f(X_t) \qquad \text{iconic} \quad X_{t+1} = RX(1-X) = RX - RX^2$$

logistic map



kein Fixpunkt

steady state $\quad X_{t+1} \overset{!}{=} X_t = X^* = X^0$

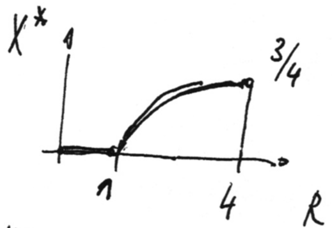
$X_1^* = 0$

X_2^*:

$$1 = R(1-X^*) = R - RX^*$$

$$X_2^* = \frac{R-1}{R} \qquad X_2^* \geq 0$$

$$= 0 \quad R \geq 1$$



bifurcation diagram

Stabilität:

kleine Auslenkung $\quad X^* + \Delta X$

$$\cancel{X^* + \Delta X = f(X^* + \Delta X)} = \cancel{f(X^*) + f'(X^*)\,\Delta X} + \dots$$

$$= \cancel{X^* + f'(X^*)\,\Delta X}$$

$$\cancel{X^* + \Delta_{t+1}}$$

$$\underset{t+1}{(X^* + \Delta_t)} : f(X^* + \Delta_t) \approx f(X^*) + f'(X^*) \cdot \Delta_t$$

$$X^* + \Delta_{t+1} = X^* + f'(X^*) \cdot \Delta_t$$

stable when $|f'| < 1$

$\underline{\Delta_{t+1} = f'(X)\,\Delta_t}$

$f'(x^*) = 2-R$

$|f'| < 1$ $\quad 1 < R < 3$ Stabilität

Differentialgleichungen:

- deterministisch ; zeitkontinuierlich
- 1st order ; d.h nur erste Ableitung

$\Rightarrow$ gewöhnliche Differentialgleichungen $\Rightarrow$ ordinary differential equation (ODE)

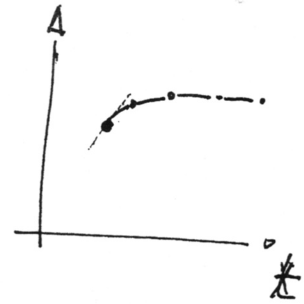
$$\frac{dx}{dt} = \dot{x} = f(x,t)$$

nicht-autonom : $\frac{dx}{dt} = f(x)$; $\dot{x} = f(x)$

Definition der Ableitung:

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = f(x)$$

Definition: Änderung von X in der Zeit



$$x(t+\Delta t) = x(t) + f(x)\cdot \Delta t$$

Eulerverfahren für Zeitentwicklung

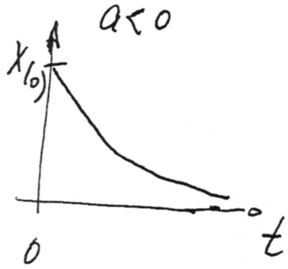
* Problem Computer

Beispiel:

lineare Differentialgleichung 1ter Ordnung

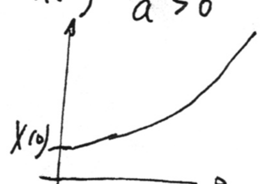
$$\frac{dx}{dt} = a \cdot x \qquad x(t) = x(0)\, e^{at}$$

$a < 0$



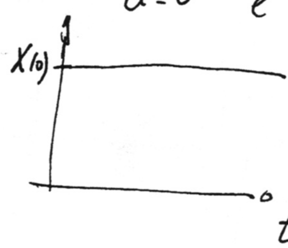
exponential decay

$a > 0$



exponential growth

$a = 0$ $\quad e^0 = 1$



constant

Referenzfall für alle ODE Gleichungen

Beispiel: ~~Verhalten~~ ~~equ~~ 1Dimensional; nichtlineare ODE

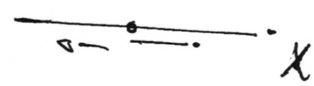
$$\frac{dx}{dt} = Kx - ax^2 = Kx\left(1 - \underbrace{\frac{a}{K}}_{\hat{=}1}x\right)$$

Wachstum von Populationen

$\hat{=}1$ → similar to logistic map

! keine oscillationen; kein Chaos !

State space



Steady state / Fixpunkt

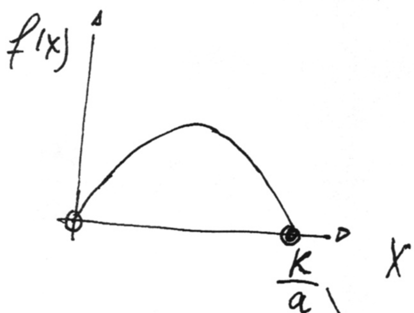
$$\frac{dx}{dt} = 0$$ keine Änderung in der Zeit (immer gleiches Vorgehen)

$$f(x) = 0$$

$$0 = Kx\left(1 - \frac{a}{K}x\right)$$

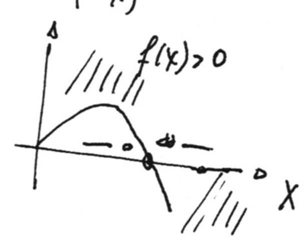
$$x_1^* = 0$$
$$x_2^* = \frac{K}{a}$$

graphische Lösung



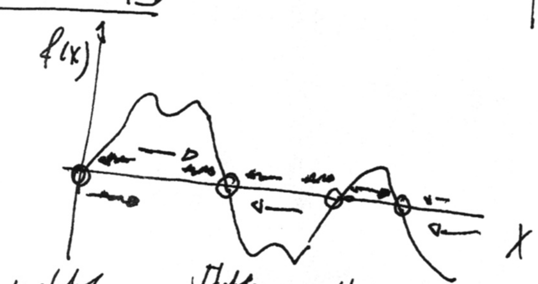
2 Fixpunkte <=> 2 Schnittpunkte

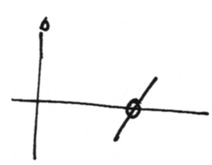
Stabilität: Änderung von x



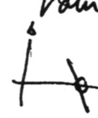
$\dot{f}(x)$: Änderung von x

generell 1D





positive slope: higher than fp: value will increase



negative slope; higher than fp value will decrease

~~also~~ mathematisch: vicinity of steady state → derivative (slope)

$$\frac{\partial f}{\partial x}\Big|_{x^*} > 0 \Rightarrow \text{instabil}$$

$$\frac{\partial f}{\partial x}\Big|_{x^*} < 0 \Rightarrow \text{stabil}$$

$$\cancel{\dot{x} = f(x)}\ f(x^* + \Delta x) \approx \underbrace{f(x^*)}_{=0} + \Delta x f'(x^*)$$

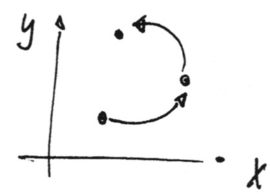
$$\frac{d}{dt}(x^* + \Delta x) = f'(x^*) \cdot \Delta x$$ back to linear: stabil $f'(x^*) < 0$ instabil $f'(x) > 0$

[Hi]gher Order ODE Models

2dim ODE

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$$

state space



steady state:
beide Variablen ändern sich nicht

$$f_x(x,y) = 0$$
$$f_y(x,y) = 0$$

im allgemeinen
nicht analytisch lösbar

We are interested in:
- trajectories
- steady states
- stability of steady states

Isoklinen: $f_x = 0$ $\quad f_y = 0$

Beispiel: Lotka Volterra Gleichung:

Predator - prey exponential growth with interactions

$$\frac{dx}{dt} = f_x(x,y) = \alpha x - \beta x \cdot y$$

$$\frac{dy}{dt} = f_y(x,y) = \gamma x \cdot y - \delta y$$

$f_x = 0 \qquad x(a - \beta y) = 0$

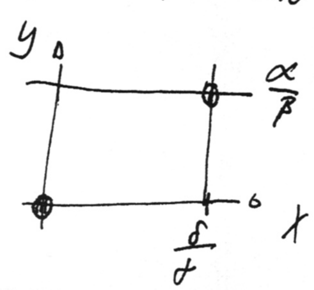
$\qquad x = 0$ oder $y = \frac{\alpha}{\beta}$

$f_y = 0 \qquad y(\gamma x - \delta) = 0$

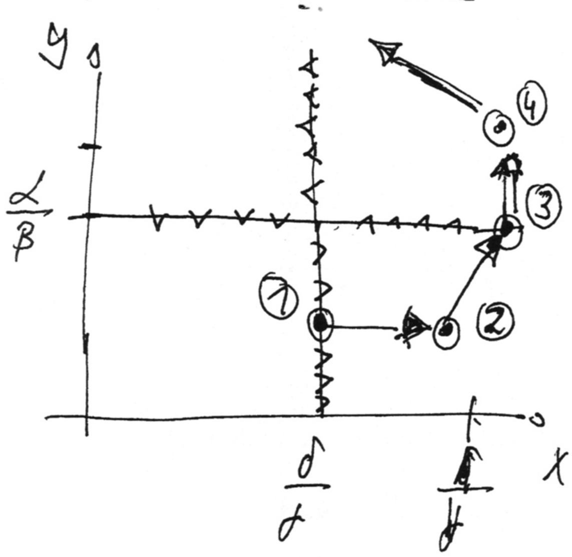
$\qquad y = 0$ oder $x = \frac{\delta}{\gamma}$

$$\left.\right\} \quad (0,0) \qquad \left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)$$

phase - plane: Nullklinen



Phasenraum / Analyse



$$f_x = \alpha x - \beta x y$$
$$f_y = \gamma x y - \delta y$$

1 Punkte auf Isoklinen

$$x = \frac{\delta}{\gamma} \quad y < \frac{\alpha}{\beta}$$

$\Rightarrow f_y = 0$

$$f_x = \alpha \frac{\delta}{\gamma} - \beta \frac{\delta}{\gamma} k_1 \frac{\alpha}{\beta} \qquad k_1 < 1$$

$$= \alpha \left(\frac{\delta}{\gamma} - k_1 \frac{\delta}{\gamma} \right) > 0$$

$\Rightarrow$ move to right

2 $x > \frac{\delta}{\gamma} \quad y < \frac{\alpha}{\beta}$

$f_x > 0$

$$f_y = \cancel{\gamma k_2 \frac{\delta}{\gamma} y - \delta y} \qquad k_2 > 1$$

$$= \gamma k_2 \frac{\delta}{\gamma} y - \delta y$$

$$= k_2 y - y > 0$$

move up, right

3 $y = \frac{\alpha}{\beta} \quad \Rightarrow f_x = 0$

$x > \frac{\delta}{\gamma} \quad f_y > 0$

move up

4 $x > \frac{\delta}{\gamma} \; ; \; y > \frac{\alpha}{\beta}$

$$f_y = \gamma k_3 \frac{\delta}{\gamma} y - \delta y \qquad k_3 > 1$$

$f_y > 0$

$$f_x = \alpha x - \beta x y$$

$$= \alpha x - \beta x k_4 \frac{\alpha}{\beta} \qquad k_4 > 1$$

$$= \alpha x (1 - k_4) < 0$$

generate nullklines first

$\Rightarrow$ oscillations

x, y predator & prey

- draw little arrows in phase plane direction of change

Numerical integration of ODE's

$$\frac{dx}{dt} = f(x)$$

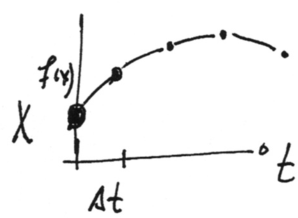
$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = f(x)$$

$$x(t+\Delta t) \approx x(t) + \Delta t\, f(x) + O(\Delta t^2)$$

Euler Method

~~Linear~~
number of steps

$$N = \frac{T}{\Delta t}$$



Error $O(\Delta t^2)$
per integration step

total error = $\cancel{N}\, O(N \cdot \Delta t^2) = O(T \cdot \Delta t)$

linear method,
decreasing error with decreasing Δt